Chapter I. Vectors and Tensors

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Abstract.

We assume the students know these materials:

Basic vector algebra: addition, subtraction, and multiplication by a scalar, linear dependence, linear independence, basis, expansion of a vector with respect to other vectors, inner product.

We will cover the topics:

Advanced vector algebra: Projection of a vector onto an axis; vector product, product of three vectors;

Brief Introduction

As human beings learned to know the natural world around them, they invented words for description, and later introduced units of measurement to quantify their description. They tried various ways, including imagination, observation, and setting up laboratories, to find out the mechanism of motion. Math is to introduce mathematical symbols (numbers, variables (length, area, volume), coordinates, functions, vectors, tensors, rate of change, equations, inequalities, etc) to model the natural phenomena. With enough symbols accumulated, a branch of math, called pure math, is devoted to the study of these symbols. The study of these symbols (rules of operations) with aims on applications to the natural world is called Applied Mathematics. A clear distinction between pure and applied mathematics is hard to draw. However, the application of mathematics is easily seen as the use of developed mathematical tools in sciences, engineering, and other fields.

Our goal will be learning the basic tools of mathematics that had and will continue to have applications. These tools will be introduced most often with some background of origin. Applications are often to follow. Our emphasis is on the math: principles and essential calculations

1.1. Vectors

Review. Vectors are

$$\mathbf{A} = (1, 1, 1), \qquad \mathbf{B} = (0, -1, 2).$$

The notation for a vector here is a bold face letter; it can be a letter with an arrow on top of it.

Scalar multiplication

$$2\mathbf{A} = (2, 2, 2).$$

Addition

$$2\mathbf{A} + \mathbf{B} = (2, 2, 2) + (0, -1, 2) = (2, 1, 4).$$

The zero vector

$$\mathbf{0} = (0, 0, 0).$$

The subtraction

$$\mathbf{A} - \mathbf{B} = (1, 2, -1).$$

Other examples

$$\mathbf{C} = (1, 2), \qquad \mathbf{D} = (0, 3).$$

And

$$2\mathbf{C} - \mathbf{D} = (2, 4) - (0, 3) = (2, 1).$$

Geometric Representation.

(Figure 1.1.1. Representation of **C**.)

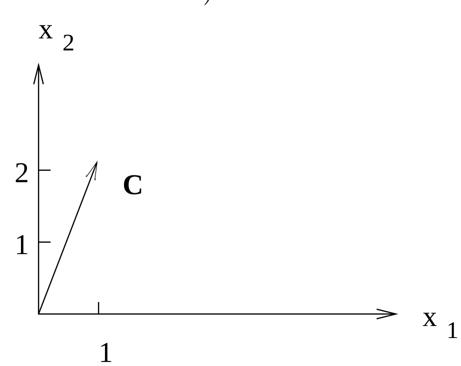

Figure 1.1.1. Representation of C.

(Figure 1.1.2. Representation of **A**) (omitted)

(Figure 1.1.3. Addition and subtraction of $\mathbf{C} + \mathbf{D}$ and $\mathbf{C} - \mathbf{D}$.)

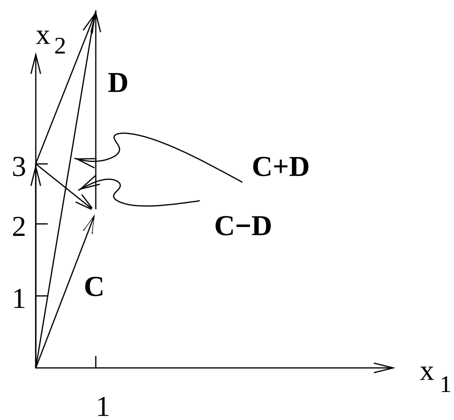

Figure 1.1.3. Representations of C+D and C−D.

(Figure 1.1.4. Scalar multiplication of **C**.) (omitted)

Length (magnitude) of a vector $\mathbf{B} = (x_1, x_2, x_3)$:

$$|\mathbf{B}| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

A **unit vector** in the direction of **B**:

$$\frac{\mathbf{B}}{|\mathbf{B}|} = \frac{(0,-1,2)}{\sqrt{0^2+(-1)^2+2^2}} = (0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}).$$

(Figure 1.1.5. A unit vector.) (omitted)

A nonzero vector **B** yields naturally an axis: the line that passes through **B** with the same direction of **B**.

(Figure 1.1.6. An axis associated with a given nonzero vector **B**.) (omitted)

Physical applications.

We can use vector **A** to represent the velocity of the water in a river and use vector **B** to represent the velocity of a boat with respect to water. Then the vector addition $\mathbf{A} + \mathbf{B}$ is the combined velocity of the boat with respect to a land-based observer.

—End of Lecture 1. —